

Asymptotic stabilizability of underactuated Hamiltonian systems with two degrees of freedom and the Lyapunov constraint based method

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Abstract

For an underactuated (simple) Hamiltonian system with two degrees of freedom, a rather general condition that ensures its stabilizability, by means of the existence of a (simple) Lyapunov function, was found in a recent paper by D.E. Chang within the context of the energy shaping method. Also, in the same paper, some additional assumptions were presented in order to ensure asymptotic stabilizability too. In this paper, in the context of the *Lyapunov constraint based method*, we show that above mentioned condition is not only a sufficient condition, but also a necessary one (to ensure stabilizability by the existence of a simple Lyapunov function). And, more importantly, we show that no additional assumption is needed to ensure asymptotic stabilizability.

1 Introduction

Consider an underactuated Hamiltonian system with two degrees of freedom. Of course, if the system is properly underactuated, then it must have exactly one actuator (otherwise, the system would be fully actuated -two actuators- or non-actuated -zero actuators-). Suppose that its related Hamiltonian function H is simple (i.e. of *kinetic-plus-potential* form) and fix a critical point of the Hamiltonian vector field of H . In Ref. [14], D.E. Chang found a sufficient condition that ensures the stabilizability¹ of such a system, by means of the existence of a (simple) Lyapunov function, at the given critical point. Concretely, fixing local coordinates (x, y, p_x, p_y) such that the actuation is along coordinate y , the critical point is $(0, 0, 0, 0)$ and H reads

$$H(x, y, p_x, p_y) = \frac{1}{2} (p_x, p_y) \begin{bmatrix} a(x, y) & b(x, y) \\ b(x, y) & c(x, y) \end{bmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} + h(x, y),$$

the mentioned condition can be written as:

$$\left(b \frac{\partial^2 h}{\partial^2 x} + c \frac{\partial^2 h}{\partial x \partial y} \right) (0, 0) \neq 0 \quad \text{or} \quad \frac{\partial^2 h}{\partial^2 x} (0, 0) > 0. \quad (1)$$

¹By *stabilizable* at a point α we mean that there exists a control law such that the related closed-loop system is stable at α .

In coordinate-free terms, according to [14], above inequalities mean that: either the linearization of the system at the given critical point is controllable, or it is uncontrollable, but with uncontrollable modes given by a purely imaginary pair.

Moreover, in the same paper, two additional conditions to ensure not only stabilizability, but also asymptotic stabilizability, were presented. All that was done within the framework of the *energy shaping method* (see for instance [3, 4, 5, 6, 7, 17, 26, 31, 34, 37]), or more precisely, within the Chang's version of the method, developed in [13, 15, 16].

In the present paper we further study the stabilizability condition (1), but in the context of the *Lyapunov constraint based (LCB) method* [19, 20, 21, 22]. In such a context, we show that:

- a. Equation (1) is not only a sufficient condition for stabilizability, but also a necessary one. More precisely, if we want to stabilize an underactuated simple Hamiltonian system (with two degrees of freedom), and to ensure such stability by exhibiting (or at least by showing the existence of) a simple Lyapunov function, then Eq. (1) must be satisfied.
- b. Equation (1) not only ensures stabilizability, but also asymptotic stabilizability. That is to say, no additional condition is needed, other than (1), in order to prove the asymptotic stabilizability for an underactuated simple Hamiltonian system with two degrees of freedom.

The paper is organized as follows. In Section §2, we make a brief review of the LCB method and its related kinetic and potential equations. We focus on the *simple* version of the method, i.e. the version that involve simple Hamiltonian and simple Lyapunov functions. Also, since we are interested in 2-dimensional configurations spaces (i.e. two degrees of freedom) and proper underactuation, we focus on systems with only one actuator. At the end of the section, and for further convenience, a new local expression of the kinetic and potential equations is presented. In Section §3, we write down such equations for Hamiltonian systems with two degrees of freedom. Then, studying the existence of their solutions, we derive the condition (1) and show that it is not only a sufficient condition for stabilizability, but also a necessary one (in the sense explained in **a**). To prove the last assertion, the *maximal character* of the LCB method is crucial (see Ref. [22]). At the end of the section, by combining the LCB method, the LaSalle's invariance principle, a Dirac-like algorithm and the Morse Lemma, we show that condition (1) also implies asymptotic stabilizability (i.e. we show the point **b**). In this point, we take advantage of the particular form that the control laws adopt, produced by the LCB method, for systems with only one actuator. Finally, we illustrate our results with an example.

We assume that the reader is familiar with basic concepts of Differential Geometry [9, 25, 30], Hamiltonian systems in the context of Geometric Mechanics [1, 2, 29], and Control Theory in a geometric language [8, 10]. Also, a basic background on stabilization of nonlinear dynamical systems is expected [24]. We will work in the C^∞ category.

Basic notation and terminology. Given an n -dimensional manifold Q , by TQ and T^*Q we denote the tangent and cotangent bundles, respectively. As it is customary, we indicate by $\langle \cdot, \cdot \rangle$ the natural pairing between $T_q Q$ and $T_q^* Q$ at every $q \in Q$. If $f : Q \rightarrow P$ is a function between two manifolds Q and P , we denote by f_* and f^* the push-forward map and its transpose, respectively.

Associated to any vector bundle on a manifold Q we have the so-called *vertical space* and the *vertical lifting map*. For the cotangent bundle, if $\pi : T^*Q \rightarrow Q$ is the canonical projection, the vertical space is the subbundle $\ker \pi_* \subset TT^*Q$ over T^*Q and the vertical lifting map is given by

$$\text{vlift}^\pi : T^*Q \times_Q T^*Q \rightarrow \ker \pi_* : (\alpha, \beta) \mapsto \text{vlift}_\alpha^\pi(\beta), \quad (2)$$

with

$$\text{vlift}_\alpha^\pi : T_\alpha^*Q \rightarrow \ker \pi_{*,\alpha} : \beta \mapsto \left. \frac{d}{dt} \right|_{t=0} (\alpha + t\beta). \quad (3)$$

Every element (resp. subbundle) of $\ker \pi_*$ is called *vertical vector* (resp. vertical subbundle). Given a function $F : T^*Q \rightarrow \mathbb{R}$, its *fiber* or *vertical derivative* $\mathbb{F}F : T^*Q \rightarrow TQ$ is defined by

$$\langle \beta, \mathbb{F}F(\alpha) \rangle = \langle dF(\alpha), \text{vlift}_\alpha^\pi(\beta) \rangle = \left. \frac{d}{dt} \right|_{t=0} F(\alpha + t\beta), \quad \forall \alpha, \beta \in T^*Q. \quad (4)$$

As usual, dF denotes the differential of F .

Consider a local chart (U, φ) of Q , with $\varphi : U \rightarrow \mathbb{R}^n$. Given $q \in U$, we write $\varphi(q) = (q^1, \dots, q^n) = \mathbf{q}$. For the induced local charts (TU, φ_*) and $(T^*U, (\varphi^*)^{-1})$ on TQ and T^*Q , respectively, we write

$$\varphi_*(v) = (q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n) = (\mathbf{q}, \dot{\mathbf{q}}), \quad (5)$$

$$(\varphi^*)^{-1}(\alpha) = (q^1, \dots, q^n, p_1, \dots, p_n) = (\mathbf{q}, \mathbf{p}),$$

or simply

$$\varphi_{*,q}(v) = \dot{\mathbf{q}} \quad \text{and} \quad (\varphi_q^*)^{-1}(\alpha) = \mathbf{p}, \quad (6)$$

for all $v \in TU$ and $\alpha \in T^*U$. On TT^*Q we shall consider the induced charts $(TT^*U, (\varphi^*)_*^{-1})$, and write

$$(\varphi^*)_*^{-1}(\varsigma) = (\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}, \dot{\mathbf{p}}), \quad (7)$$

for all $\varsigma \in TT^*U$.

2 The Lyapunov constraint based method

In this section, we make a brief review of the *Lyapunov constraint based method* (LCB method); which was first introduced in [19, 20] and further developed in [21] and [22] (see also [12] for an extension to systems with impulsive effects). As any constraint based stabilization method [11, 18, 27, 28, 32, 33, 35], the key idea is to build up a control law, which stabilizes a given actuated (or open-loop) system, as the constraint force implementing some (appropriately chosen) set of kinematic constraints. In the case of the LCB method, such constraint is the so-called *Lyapunov constraint*. Concretely, the purpose of the method is to construct, for a given underactuated Hamiltonian system, a control law and a Lyapunov function V for the resulting closed-loop system.² Such a control law is, of course, the constraint force implementing a given Lyapunov constraint. To find this force, a set of partial differential equations (PDEs), which has V among their unknowns, must be solved. We focus on underactuated systems defined by simple Hamiltonian functions (i.e. of *kinetic-plus-potential* form) and with only one actuator. This gives rise to a special version of the LCB method, the *simple LCB method*. In this version, the related set of PDEs decomposed into two subsets: the *kinetic* and *potential* equations, and the formula for the control law adopts a very simple form. At the end of the section we derive a new local expression of the kinetic and potential equations that will be used in the rest of the paper.

2.1 Stabilization and the Lyapunov constraint

Let us consider a dynamical system defined by a smooth vector field X on some manifold P . Consider also a critical point $\alpha_0 \in P$ of X . It is well-known [24] that such a system is stable at α_0 if there exists a smooth function $V : P \rightarrow \mathbb{R}$ satisfying:

²For the underactuated system, the function V represents, in essence, a *control Lyapunov function* [36].

L1 V is positive-definite w.r.t. α_0 (i.e. $V(\alpha_0) = 0$ and $V(\alpha) > 0$ for all $\alpha \neq \alpha_0$);

L2 $\langle dV(\alpha), X(\alpha) \rangle \leq 0$ for all α .

In this case, we say that V is a *Lyapunov function* for X and α_0 . It is clear that, given a trajectory $\Gamma : I \subset \mathbb{R} \rightarrow P$ of this system, condition **L2** implies that $\langle dV(\Gamma(t)), X(\Gamma(t)) \rangle = -\mu(\Gamma(t))$, for all $t \in I$, or equivalently,

$$\langle dV(\Gamma(t)), \Gamma'(t) \rangle = -\mu(\Gamma(t)), \quad (8)$$

where $\mu : P \rightarrow \mathbb{R}$ is given by $\mu(\alpha) := -\langle dV(\alpha), X(\alpha) \rangle$, for all $\alpha \in P$.

Remark 1 Observe that $\mu^{-1}(0)$ is the LaSalle surface related to V (see [24]), and $\alpha_0 \in \mu^{-1}(0)$.

That is, condition **L2** may be interpreted as a kinematic constraint on the system. Hence, roughly speaking, if we want to stabilize a dynamical system, we can think of imposing a constraint of the form (8) for appropriate functions V and μ , and then look for the constraint force that implement the mentioned constraint. We shall call it **Lyapunov constraint**. Of course, depending on the properties of V and μ , we shall have different stability characters for the point α_0 . For instance, if (besides condition **L1** for V) we ask μ to be such that $\{\alpha_0\}$ is the bigger invariant subset of $\mu^{-1}(0)$, the *LaSalle's invariance principle* would ensure (local) asymptotic stability for α_0 . This is true, for example, if we assume that property **L1** also holds for μ , what would imply that

$$\langle dV(\alpha), X(\alpha) \rangle < 0 \quad \text{for all } \alpha \neq \alpha_0.$$

If in addition we ask V to be a proper function (and P to be connected), then global asymptotic stability for α_0 would be ensured. (For a proof of these results, see [24] again).

Now, let us focus our attention on Hamiltonian systems. Take $P = T^*Q$ for some Q , fix a smooth function $H : T^*Q \rightarrow \mathbb{R}$ and consider the Hamiltonian system on Q defined by H . Given two functions $V, \mu : T^*Q \rightarrow \mathbb{R}$, let us impose the constraint (8) on this system. In other words, let us define the submanifold

$$\mathcal{P} := \bigcup_{\alpha \in T^*Q} \{\varsigma \in T_\alpha T^*Q : \langle dV(\alpha), \varsigma \rangle = -\mu(\alpha)\} \subset TT^*Q, \quad (9)$$

and impose the condition $\Gamma'(t) \in \mathcal{P}$ on the trajectories.

Remark 2 Notice that, if (U, φ) is a coordinate chart of Q , in terms of the induced chart on TT^*Q [see Eq. (7)] the submanifold \mathcal{P} is locally given by the equation³

$$\frac{\partial V}{\partial q^i}(\mathbf{q}, \mathbf{p}) \dot{q}^i + \frac{\partial V}{\partial p_i}(\mathbf{q}, \mathbf{p}) \dot{p}_i = -\mu(\mathbf{q}, \mathbf{p}),$$

where, for simplicity, φ is omitted in the above expression. So, \mathcal{P} defines a first order constraint in positions and momenta, or equivalently (since the momentum is, in essence, the derivative of the position) a second order constraint in positions.

Suppose that we want to implement this constraint by actuators exerting forces lying inside a vertical subbundle $\mathcal{W} \subset TT^*Q$ (recall that a force in the Hamiltonian formalism is a vertical vector). Observe that the pair (H, \mathcal{W}) defines an underactuated system and the triple $(H, \mathcal{P}, \mathcal{W})$ defines a *second order constrained system* [19]. The trajectories for the latter are the integral curves of the vector fields $X \in \mathfrak{X}(T^*Q)$ such that

$$X \subset \mathcal{P} \quad \text{and} \quad X - X_H \subset \mathcal{W},$$

³From now on, sum over repeated indices convention is assumed; i.e. an expression of the form $A_i B^i$ will mean $\sum_i A_i B^i$, unless we say the contrary.

or equivalently, the integral curves of the vector fields $X_H + Y$, being $Y \in \mathfrak{X}(T^*Q)$ such that

$$X_H + Y \subset \mathcal{P} \quad \text{and} \quad Y \subset \mathcal{W}. \quad (10)$$

As usual, X_H denotes the Hamiltonian vector field associated to H (w.r.t. the canonical symplectic form on T^*Q). Each vertical vector field Y represents a *constraint force* that implements the constraint under consideration. According to (9) and (10), Y must satisfy

$$\langle dV(\alpha), X_H(\alpha) + Y(\alpha) \rangle = -\mu(\alpha) \quad \text{and} \quad Y(\alpha) \in \mathcal{W}, \quad (11)$$

for all $\alpha \in T^*Q$. Suppose now that, for some critical point $\alpha_0 \in T^*Q$ of X_H , the condition **L1** holds for the function V . Also, suppose that μ is non-negative. Then, if Y is a solution of the above equation such that $Y(\alpha_0) = 0$, the function V is a Lyapunov function for $X = X_H + Y$ and α_0 . In fact, α_0 is a critical point for X and V satisfies **L1** and **L2**. As a consequence, α_0 is a stable point for the dynamical system defined by X . Of course, if stronger conditions are imposed on V and μ (as discussed at the beginning of this section), stronger assertions about the stability of X around α_0 can be made.

Remark 3 *If a solution Y of (11) exists along an open subset $\pi^{-1}(U) = T^*U \subset T^*Q$ (resp. an open subset $T \subset T^*Q$) containing α_0 , namely a local solution of (11), the same assertions can be made, just replacing Q by U (resp. T^*Q by T).*

Note that the role of Y is two-fold:

- a constraint force for the constrained system $(H, \mathcal{P}, \mathcal{W})$;
- a control law for the underactuated system (H, \mathcal{W}) .

The previous discussion drives us to the following method for (asymptotic) stabilization of underactuated Hamiltonian systems.

Definition 4 *Given an underactuated system (H, \mathcal{W}) on Q and a critical point $\alpha_0 \in T^*Q$ of X_H , the **Lyapunov constraint based (LCB) method** consists in finding two functions $V, \mu : T^*Q \rightarrow \mathbb{R}$ and a vector field Y on T^*Q such that: V is positive definite w.r.t. α_0 , μ is non-negative, $Y(\alpha_0) = 0$ and Eq. (11) is satisfied.*

Thus, given (H, \mathcal{W}) and α_0 as above, this method gives rise to a closed-loop mechanical system with vector field $X := X_H + Y$, and a Lyapunov function V for X and α_0 . It was shown in [22] that any method of this kind, i.e. any method that produces a control law and a Lyapunov function for its related closed-loop system, is *included* in the LCB method. More precisely, the set of control laws produced by any of these methods, for each pair (H, \mathcal{W}) and each point α_0 , is contained in the corresponding set of the LCB method. In particular, the LCB method includes every version of the *energy shaping method* [3, 4, 5, 6, 7, 17, 26, 31, 34, 37]. This grants to the LCB method a *maximal* character that we will exploit later.

2.2 The simple LCB method for systems with one actuator

Let us study the Eq. (11). Since we are interested in underactuated Hamiltonian systems (H, \mathcal{W}) with two degrees of freedom, i.e. those for which $\dim Q = 2$, then the vertical distribution \mathcal{W} can only have rank 1 (otherwise, we would have a fully actuated or a non-actuated system). So, we shall assume from now on that:

W1 $\mathcal{W}_\alpha = \langle \Omega(\alpha) \rangle$, where⁴ Ω is a non-vanishing vertical vector field on T^*Q .

⁴Given a vector space V , we use $\langle v_1, \dots, v_n \rangle$ to denote the linear subspace spanned by vectors $v_1, \dots, v_n \in V$.

W2 $\Omega(\alpha) = \text{vlift}_\alpha^\pi(\xi(\pi(\alpha)))$ [recall Eqs. (2) and (3)], with $\xi : Q \rightarrow T^*Q$ a differential form.

Above conditions say that \mathcal{W} can be defined by a rank 1 subbundle $W \subset T^*Q$. More precisely, we have that

$$\mathcal{W}_\alpha = \text{vlift}_\alpha^\pi(W_{\pi(\alpha)}), \quad \text{where } W_q = \langle \xi(q) \rangle \subset T_q^*Q.$$

Let us also assume that H is *simple*, i.e.

$$H(\alpha) = \frac{1}{2} \langle \alpha, \rho^\sharp(\alpha) \rangle + h(q),$$

where $q = \pi(\alpha)$, $h \in C^\infty(Q)$ and ρ is a Riemannian metric on Q . [As usual, $\rho^\sharp : T^*Q \rightarrow TQ$ denotes the inverse of the map $v \in TQ \mapsto \rho(v, \cdot) \in T^*Q$]. The functions ρ and h defines the *kinetic* and *potential* terms of H , respectively. Under these conditions, we have the next result (see Ref. [21]).

Theorem 5 *Given (H, \mathcal{W}) , where H is simple and \mathcal{W} satisfies **W1** and **W2**, there exists a solution Y of (11), with V simple and μ non-negative, if and only if⁵*

$$\{V, H\}(\mathbb{F}V^{-1}(v)) = 0, \quad \forall v \in W^0. \quad (12)$$

The solution exists for every (non-negative) function μ such that

$$\frac{\mu}{\langle \xi(\pi(\cdot)), \mathbb{F}V(\cdot) \rangle} \in C^\infty(T^*Q), \quad (13)$$

and it is univocally given by the formula

$$Y(\alpha) := \lambda(\alpha) \text{vlift}_\alpha^\pi(\xi(\pi(\alpha))), \quad (14)$$

with

$$\lambda(\alpha) := -\frac{\mu(\alpha) + \{V, H\}(\alpha)}{\langle \xi(\pi(\alpha)), \mathbb{F}V(\alpha) \rangle}. \quad (15)$$

To fulfill the Equation (13), we can choose

$$\mu(\cdot) = \varkappa \langle \xi(\pi(\cdot)), \mathbb{F}V(\cdot) \rangle^2, \quad \text{with } \varkappa > 0. \quad (16)$$

So, the relevant condition for finding a solution of (11) is given by the Eq. (12), which is a system of PDEs with V as its unique unknown. Let us re-write these PDEs in local terms. Note first that, since V is simple, then we can write

$$V(\alpha) = \frac{1}{2} \langle \alpha, \phi^\sharp(\alpha) \rangle + v(q)$$

for some function $v \in C^\infty(Q)$ and some Riemannian metric ϕ on Q . Choosing a coordinate chart (U, φ) on Q [see Eqs. (5) and (6)], omitting φ we have

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2} p_i \mathbb{H}^{ij}(\mathbf{q}) p_j + h(\mathbf{q}) \quad (17)$$

and

$$V(\mathbf{q}, \mathbf{p}) = \frac{1}{2} p_i \mathbb{V}^{ij}(\mathbf{q}) p_j + v(\mathbf{q}). \quad (18)$$

Here, \mathbb{H} and \mathbb{V} stand for the coordinate matrix representation of the Riemannian metrics ρ and ϕ . More precisely, if we define

$$\mathbb{M}^{ij} := \rho \left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j} \right) \quad \text{and} \quad \mathbb{N}^{ij} := \phi \left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j} \right),$$

⁵Here, $\{\cdot, \cdot\}$ denotes the canonical Poisson bracket and $\mathbb{F}V$ is the fiber derivative of V [see Eq. (4)].

then $\mathbb{H} = \mathbb{M}^{-1}$ and $\mathbb{V} = \mathbb{N}^{-1}$. In these terms, Eq. (12) translates to the following PDEs:

$$\left(\frac{\partial \mathbb{V}^{ij}}{\partial q^k}(\mathbf{q}) \mathbb{H}^{kl}(\mathbf{q}) - \frac{\partial \mathbb{H}^{ij}}{\partial q^k}(\mathbf{q}) \mathbb{V}^{kl}(\mathbf{q}) \right) p_i p_j p_l = 0, \quad (19)$$

the *kinetic equations*, and

$$\left(\frac{\partial v}{\partial q^k}(\mathbf{q}) \mathbb{H}^{kl}(\mathbf{q}) - \frac{\partial h}{\partial q^k}(\mathbf{q}) \mathbb{V}^{kl}(\mathbf{q}) \right) p_l = 0, \quad (20)$$

the *potential equations*. They must be satisfied for all (\mathbf{q}, \mathbf{p}) such that

$$\mathbf{p}^t \mathbb{V}(\mathbf{q}) \varphi_q^{*-1}(\xi(q)) = 0. \quad (21)$$

For the last equation, we are using that $\mathbb{F}V = \phi^\sharp$, and consequently the local representation of $\mathbb{F}V$ is given by the matrix \mathbb{V} .

Now, fix a critical point α_0 of X_H . Since H is simple, it can be shown that α_0 is of the form $(q_0, 0)$ (i.e. α_0 belongs to the null subbundle of T^*Q), being $q_0 \in Q$ a critical point of h . In this case, it is clear that V is positive-definite w.r.t. α_0 if and only if so is v w.r.t. q_0 .

All these considerations (including the Theorem 5), combined with the Definition 4, give rise to the following stabilization method.

Definition 6 *Given an underactuated system (H, \mathcal{W}) as above, and given a critical point $\alpha_0 = (q_0, 0) \in T^*Q$ of X_H , the **(local) simple LCB method** consists in:*

1. *fixing a local chart (U, φ) around q_0 ;*
2. *finding a set of functions $\mathbb{V}^{ij}, v : \varphi(U) \rightarrow \mathbb{R}$ satisfying (19), (20) and (21), such that*
 - (a) *the numbers $\mathbb{V}^{ij}(\mathbf{q})$ define the coefficients of a symmetric and positive-definite matrix for all $\mathbf{q} \in \varphi(U)$,*
 - (b) *and the function v is positive-definite w.r.t. $\mathbf{q}_0 := \varphi(q_0)$;*
3. *defining a (local) vector field Y by Equations (14) and (15), with μ satisfying (13) and V given by Equation (18) and the last point.*

This is a local version of the simple LCB method presented in [22] (see also [21]). Nevertheless, the word “local” will be omitted, just for brevity. It is clear that a *maximality* property is also fulfilled by this method. More precisely, we have the next result.

Theorem 7 *Consider an underactuated system (H, \mathcal{W}) on Q , with H simple and \mathcal{W} satisfying **W1** and **W2**, and consider a critical point $\alpha_0 = (q_0, 0) \in T^*Q$ of X_H . Then, there exists a vector field Y with image inside \mathcal{W} and a simple Lyapunov function V for $X_H + Y$ and α_0 , both Y and V defined at least on some open subset U containing q_0 , if and only if Y and V are given by the (local) simple LCB method.*

Proof. If there exists a vector field Y with image inside \mathcal{W} and a Lyapunov function V for $X_H + Y$ and α_0 , it is clear that Y and V must satisfy (11) for some non-negative function μ . If V is simple, according to Theorem 5, V must satisfy (12) and Y must be given by (15) and (14). Since the local expression of (12) (in every local chart) is given precisely by Eqs. (19), (20) and (21), the vector field Y and the function V must be given around q_0 as in Definition 6, i.e. by the (local) simple LCB method.

The converse is precisely the purpose of the method. \square

It was shown in [22] that Eqs. (19) and (20) are exactly the *matching conditions* corresponding to the Chang’s version of the energy shaping method [13, 15, 16]. Moreover, it was shown in the same paper that the simple LCB method and the Chang’s version of the energy shaping method are equivalent stabilization methods, in the sense that they give rise to the same sets of control laws. In spite of this equivalence, in this paper we prefer to use the *language* of LCB method, mainly because of the formula that defines its control laws [see Eqs. (14) and (15)], which will be very useful in the last section of the paper.

2.3 Kinetic and potential equations for *integrable actuators*

In this subsection, we derive a new expression for the Eqs. (19), (20) and (21) in the case in which the subbundle $W \subset TQ$, spanned by ξ , is integrable.⁶

Remark 8 *Note that, if $\dim Q = 2$, then every rank 1 subbundle W must be integrable.*

For such subbundles, it is easy to show that there exists a coordinate chart (U, φ) of Q such that

$$\xi(q) = dq^n|_q = \varphi_q^* \left(\underbrace{0, \dots, 0}_{n-1 \text{ times}}, 1 \right), \quad \forall q \in U, \quad (22)$$

where $n = \dim Q$. We shall say that (U, φ) is **adapted** to W . Using this chart,

$$\text{vlift}_\alpha^\pi(\xi(\pi(\alpha))) = \frac{\partial}{\partial p_n} \Big|_\alpha, \quad \forall \alpha \in T^*U,$$

and accordingly [see Eq. (14)]

$$Y(\alpha) = \lambda(\alpha) \frac{\partial}{\partial p_n} \Big|_\alpha \quad (23)$$

along T^*U . Then, Eq. (21) translates to

$$p_i \mathbb{V}^{ni}(\mathbf{q}) = 0. \quad (24)$$

On the other hand, since we are interested in functions \mathbb{V}^{ij} that define a symmetric and positive-definite matrix (see Definition 6), each diagonal coefficient \mathbb{V}^{ii} must be strictly positive, and consequently non null. Thus, Eq. (24) implies that

$$p_n = -\frac{\mathbb{V}^{n\sigma}(\mathbf{q})}{\mathbb{V}^{nn}(\mathbf{q})} p_\sigma. \quad (25)$$

We are using greek indices running from 1 to $n-1$ and latin indices running from 1 to n .

2.3.1 The kinetic equation

Now, let us write the kinetic equation (19) in an adapted chart. Using (25), if we write (19) as (omitting \mathbf{q} for brevity)

$$\left(\frac{\partial \mathbb{V}^{ij}}{\partial q^k} p_i p_j \right) (\mathbb{H}^{kl} p_l) = \left(\frac{\partial \mathbb{H}^{ij}}{\partial q^k} p_i p_j \right) (\mathbb{V}^{kl} p_l),$$

the left hand side takes the form

$$\left(\frac{\partial \mathbb{V}^{ij}}{\partial q^k} p_i p_j \right) (\mathbb{H}^{kl} p_l) = \frac{1}{(\mathbb{V}^{nn})^2} \left[p_i \left(\mathbb{V}^{nn} \frac{\partial \mathbb{V}^{i\mu}}{\partial q^k} - \frac{\partial \mathbb{V}^{in}}{\partial q^k} \mathbb{V}^{n\mu} \right) \right] (\mathbb{V}^{nn} \mathbb{H}^{k\sigma} - \mathbb{H}^{kn} \mathbb{V}^{n\sigma}) p_\sigma p_\mu.$$

(Recall that $\mathbb{V}^{ij} = \mathbb{V}^{ji}$ and the same is true for \mathbb{H}). To further simplify this expression, we define

$$\delta^{\mu\nu} := \frac{\mathbb{V}^{\mu\nu} \mathbb{V}^{nn} - \mathbb{V}^{n\mu} \mathbb{V}^{n\nu}}{\mathbb{V}^{nn}} \quad \text{and} \quad \gamma^\mu := \frac{\mathbb{V}^{n\mu}}{\mathbb{V}^{nn}}. \quad (26)$$

Remark 9 *Note that, in these terms, the local version of (16) reads*

$$\mu(\mathbf{q}, \mathbf{p}) = \varkappa \left(p_i \mathbb{V}^{ni}(\mathbf{q}) \right)^2 = \varkappa \left(\gamma^\sigma(\mathbf{q}) p_\sigma + p_n \right)^2 \left(\mathbb{V}^{nn}(\mathbf{q}) \right)^2. \quad (27)$$

Using the new variables, we have that

$$\left(\frac{\partial \mathbb{V}^{ij}}{\partial q^k} p_i p_j \right) (\mathbb{H}^{kl} p_l) = \frac{\partial \delta^{\mu\nu}}{\partial q^k} (\mathbb{H}^{k\sigma} - \mathbb{H}^{kn} \gamma^\sigma) p_\mu p_\nu p_\sigma.$$

⁶We say that a subbundle $W \subset T^*Q$, spanned by differential forms $\alpha^1, \dots, \alpha^m$, is integrable if and only if the algebraic ideal \mathcal{I} generated by $\alpha^1, \dots, \alpha^m$ is also a differential ideal, i.e. $d\mathcal{I} \subset \mathcal{I}$.

We can work the right hand side in a similar fashion to get

$$\left(\frac{\partial \mathbb{H}^{ij}}{\partial q^k} p_i p_j \right) (\mathbb{V}^{kl} p_l) = \left(\frac{\partial \mathbb{H}^{\mu\nu}}{\partial q^\tau} - \frac{\partial \mathbb{H}^{n\mu}}{\partial q^\tau} \gamma^\nu - \gamma^\mu \frac{\partial \mathbb{H}^{n\nu}}{\partial q^\tau} + \gamma^\mu \gamma^\nu \frac{\partial \mathbb{H}^{nn}}{\partial q^\tau} \right) \delta^{\tau\sigma} p_\mu p_\nu p_\sigma.$$

Then, defining

$$A^{\sigma k} := \mathbb{H}^{\sigma k} - \mathbb{H}^{nk} \gamma^\sigma \quad (28)$$

and

$$B_\tau^{\mu\nu} := \frac{\partial \mathbb{H}^{\mu\nu}}{\partial q^\tau} - \frac{\partial \mathbb{H}^{n\mu}}{\partial q^\tau} \gamma^\nu - \gamma^\mu \frac{\partial \mathbb{H}^{n\nu}}{\partial q^\tau} + \gamma^\mu \gamma^\nu \frac{\partial \mathbb{H}^{nn}}{\partial q^\tau}, \quad (29)$$

the kinetic equation reads

$$\left(A^{\sigma k} \frac{\partial \delta^{\mu\nu}}{\partial q^k} - B_\tau^{\mu\nu} \delta^{\tau\sigma} \right) p_\mu p_\nu p_\sigma = 0, \quad (30)$$

for all n -tuples $(p_1, \dots, p_{n-1}, p_n = -\gamma^\mu p_\mu)$.

Remark 10 *It is clear that the map*

$$\mathbb{V}^{ij} \mapsto (\delta^{\mu\nu}, \gamma^\mu, \mathbb{V}^{nn}) \quad (31)$$

defined by (26), with \mathbb{V}^{ij} symmetric and $\mathbb{V}^{nn} \neq 0$, is smooth onto its image and with smooth inverse given by

$$\mathbb{V}^{n\mu} = \mathbb{V}^{\mu n} = \gamma^\mu \mathbb{V}^{nn} \quad \text{and} \quad \mathbb{V}^{\mu\nu} = \delta^{\mu\nu} + \gamma^\mu \gamma^\nu \mathbb{V}^{nn}. \quad (32)$$

The left hand side of (30) is a polynomial in the $n-1$ variables p_μ whose coefficients are given by the symmetrization of the term

$$A^{\sigma k} \frac{\partial \delta^{\mu\nu}}{\partial q^k} - B_\tau^{\mu\nu} \delta^{\tau\sigma}$$

with respect to the indices $\mu\nu\sigma$. Since the functions $\delta^{\mu\nu}$ and $B_\tau^{\mu\nu}$ are symmetric in $\mu\nu$ [see (26) and (29)], we can consider only cyclic permutations in the mentioned symmetrization. Thus, in order to fulfill (30), for each choice of indices (μ, ν, σ) we must have

$$\sum_{(\mu\nu\sigma)} \left(A^{\sigma k} \frac{\partial \delta^{\mu\nu}}{\partial q^k} - B_\tau^{\mu\nu} \delta^{\tau\sigma} \right) = 0, \quad (33)$$

where $\sum_{(\mu\nu\sigma)}$ stands for the sum over cyclic permutations of μ, ν and σ . Observe that \mathbb{V}^{nn} does not appear above, so, the unknowns of Eq. (33) are actually the functions $\delta^{\mu\nu}$ and γ^μ only. Moreover, since just the derivatives of the functions $\delta^{\mu\nu}$ (up to order one) appear in (33), we can see the latter as a system of first order PDEs for $\delta^{\mu\nu}$, with unknown “parameters” γ^μ .

2.3.2 The potential equation

In the case of (20), we can proceed analogously to find the equation

$$A^{\sigma k} \frac{\partial v}{\partial q^k} - \frac{\partial h}{\partial q^\tau} \delta^{\tau\sigma} = 0, \quad (34)$$

which is a system of $n-1$ first order PDEs for v , with unknown “parameters” $\delta^{\mu\nu}$ and γ^μ , with $\mu, \nu = 1, \dots, n-1$.

Remark 11 *Observe that the set of PDEs given by (33) and (34) forms a **linear** system of PDEs for the unknowns $\delta^{\mu\nu}$ and v . The unknowns γ^μ can be seen again as parameters.*

Summarizing, in the case in which W is an integrable (rank 1) subbundle and we consider a coordinate chart adapted to W , the kinetic and potential equations (19) and (20) [together with (21)] are given by (33) and (34), respectively. In order to go back from variables $\delta^{\mu\nu}$, γ^μ and \mathbb{V}^{nn} to the original variables \mathbb{V}^{ij} , we just need to consider the Eq. (32).

3 Stabilizability of bidimensional systems

Given an underactuated Hamiltonian system (H, \mathcal{W}) as above, but on a 2-manifold Q , and given a critical point α_0 of its related Hamiltonian vector field X_H , in this section we shall find a necessary and sufficient condition to ensure that such a system is stabilizable at α_0 by means of the existence of a simple Lyapunov function. Moreover, at the end of this section we show that the mentioned condition not only ensure stabilizability, but also (local) asymptotic stabilizability. It is worth mentioning that, in the present context, by “(asymptotically) stabilizable at α_0 ” we mean that: there exists a control law $Y \subset \mathcal{W}$ such that the related closed-loop system $X_H + Y$ is (asymptotically) stable at α_0 .

We shall derive above mentioned results by using the simple LCB method presented in the previous section.

3.1 The kinetic and potential equations for bidimensional systems

In this section we shall write the Eqs. (33) and (34) when $n = 2$ (see Remark 8). Also, we shall analyze in this case the positivity conditions of the simple LCB method (recall point 2 of Definition 6).

In order to simplify the notation, define

$$x := q^1, \quad y := q^2, \quad (35)$$

and

$$\mathbb{H}(x, y) =: \begin{bmatrix} a(x, y) & b(x, y) \\ b(x, y) & c(x, y) \end{bmatrix}, \quad \mathbb{V}(x, y) =: \begin{bmatrix} f(x, y) & g(x, y) \\ g(x, y) & l(x, y) \end{bmatrix}. \quad (36)$$

Note that, since \mathbb{H} is positive-definite, $a(x, y), c(x, y) > 0$ and $\Delta(x, y) := a(x, y)c(x, y) - b^2(x, y) > 0$. Also, since the method looks for a positive-definite matrix \mathbb{V} , we must ask

$$f(x, y) > 0 \quad \text{and} \quad f(x, y)l(x, y) - g^2(x, y) > 0, \quad (37)$$

which imply that $l(x, y) > 0$. To further simplify the notation, and recalling Equations (26), (28) and (29), define

$$\delta := \delta^{11} = \frac{fl - g^2}{l}, \quad \gamma := \gamma^1 = \frac{g}{l}, \quad (38)$$

and

$$A := (A^{11}, A^{12}) = (a - b\gamma, b - c\gamma), \quad B := B_1^{11} = a_x - 2\gamma b_x + \gamma^2 c_x. \quad (39)$$

(From now on, the subindices x and y denote partial differentiation). Note that, since $\mathbb{V}^{22} = l$, the change of variables $\mathbb{V}^{ij} \mapsto (\delta^{\mu\nu}, \gamma^\mu, \mathbb{V}^{nn})$ of the general case [see the Eqs. (26) and (31)] reduces to $\mathbb{V}^{ij} \mapsto (\delta, \gamma, l)$. In terms of these new variables, the kinetic equation [see (33)] and the positivity conditions (37) say that $l > 0$ and

$$(a - b\gamma)\delta_x + (b - c\gamma)\delta_y = B\delta, \quad \delta > 0, \quad (40)$$

while the potential equation (34) adopt the form

$$(a - b\gamma)v_x + (b - c\gamma)v_y = h_x \delta. \quad (41)$$

Now, assume that a critical point $(q_0, 0)$ of X_H is fixed.

Remark 12 Recall that, if $(q_0, 0)$ is critical for X_H , then q_0 is critical for the function h .

For simplicity, suppose that the local chart we are using (U, φ) is centered at q_0 , i.e. $\varphi(q_0) = (0, 0)$ (this always can be done, without disturbing the condition (22)). Then, the positivity condition for v read

$$v(0, 0) = 0, \quad v(x, y) > 0 \quad \text{for all} \quad (x, y) \neq (0, 0). \quad (42)$$

Summing up, for two-dimensional systems, the point 2 of the simple LCB method is condensed in Equations (40), (41) and (42) for the unknowns (δ, γ, v) , and the condition $l > 0$.

3.2 A sufficient condition for stabilizability

Let us continue with the underactuated system (H, \mathcal{W}) , the critical point $(q_0, 0)$ of X_H , and the notation of the previous subsection. If we find a solution to the Equations (40), (41) and (42), then, by applying the step 3 of the simple LCB method, we can construct a vector field Y and a simple Lyapunov function for $X_H + Y$ and $(q_0, 0)$ [at least around $(q_0, 0)$]. This means that (H, \mathcal{W}) is stabilizable around $(q_0, 0)$. So, the stabilizability of (H, \mathcal{W}) around $(q_0, 0)$ can be analyzed by studying the existence of solutions (δ, γ, v) of Equations (40), (41) and (42). To do that, let us assume again that we are working with an adapted chart (U, φ) centered at q_0 , i.e.

$$\mathbf{q}_0 := \varphi(q_0) = (0, 0) =: \mathbf{0},$$

and let us consider the next two lemmas.

Lemma 13 *Given a function γ satisfying*

$$\gamma(\mathbf{0}) \neq \frac{b(\mathbf{0})}{c(\mathbf{0})} \quad (43)$$

and

$$[(a - b\gamma)h_{xx} + (b - c\gamma)h_{xy}](\mathbf{0}) > 0, \quad (44)$$

there exist functions δ and v such that (δ, γ, v) is a solution of (40), (41) and (42).

Proof. Let us begin with (40). This is a first-order PDE, so we can use the *Method of Characteristics* to find a solution around $\mathbf{0}$. But, in order for this to make sense, we need a suitable boundary condition on a non-characteristic submanifold Γ . Observe that the characteristic vector field is $A = (a - b\gamma, b - c\gamma)$. Then, we may take the submanifold $\Gamma \subset \mathbb{R}^2$ to be the x -axis, i.e. to take

$$\Gamma = \{(x, 0) : x \in \mathbb{R}\} \cap \varphi(U),$$

so long as we ensure that the second component of A is nonzero around $\mathbf{0}$. But this amounts to choose γ such that (43) holds. Since we need that $\delta > 0$, we can impose the boundary condition $\delta|_{\Gamma} = s$, where $s : \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $s(0) > 0$. In this case, the Theorem of Characteristics states that there is a unique solution δ of (40) such that $\delta(x, 0) = s(x)$, which implies, by continuity, that $\delta(x, y) > 0$ around $\mathbf{0}$. We can shrink U , if necessary, in order to ensure that $\delta > 0$ along all of U . From now on, we shall use this shrinking process implicitly (finitely many times).

Let us continue with (41) and (42). The former is also a first-order PDE, and with the same characteristic vector field A . Assuming (43) again, the x -axis is a non-characteristic submanifold and we can impose $v|_{\Gamma} = r$, where $r : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function such that $r(0) = 0$. This implies that $v(\mathbf{0}) = 0$, which is the first part of (42). The second part says that $\mathbf{0}$ is an isolated minimum for v , or equivalently, $\mathbf{0}$ is critical for v and the Hessian of v is positive-definite at $\mathbf{0}$. Let us analyze these conditions. Since $\mathbf{0}$ is critical for h (see Remark 12), i.e. $(h_x(\mathbf{0}), h_y(\mathbf{0})) = \mathbf{0}$, it follows from (41) that v must satisfy

$$[(a - b\gamma)v_x + (b - c\gamma)v_y](\mathbf{0}) = 0.$$

Thus, since $b(\mathbf{0}) - c(\mathbf{0})\gamma(\mathbf{0}) \neq 0$ [see (43)], in order to have that $(v_x(\mathbf{0}), v_y(\mathbf{0})) = \mathbf{0}$, it suffices to ask $v_x(\mathbf{0}) = r'(0) = 0$. By now, we have that r and s must satisfy

$$s(0) > 0, \quad r(0) = r'(0) = 0. \quad (45)$$

On the other hand, the Hessian of v is positive-definite at $\mathbf{0}$ if and only if

$$v_{xx}(\mathbf{0}) > 0 \quad \text{and} \quad (v_{xx}v_{yy} - v_{xy}^2)(\mathbf{0}) > 0. \quad (46)$$

It is easy to compute the second partial derivatives of v at $\mathbf{0}$ using (41) and the boundary conditions $\delta|_{\Gamma} = s$ and $v|_{\Gamma} = r$. This gives, omitting the evaluation point $\mathbf{0}$,

$$\begin{aligned} v_{xx} &= r''(0), \\ v_{xy} &= \frac{h_{xx} s(0) - (a - b\gamma) r''(0)}{(b - c\gamma)}, \\ v_{yy} &= \frac{h_{xy} s(0) (b - c\gamma) - (a - b\gamma) h_{xx} s(0) + (a - b\gamma)^2 r''(0)}{(b - c\gamma)^2}. \end{aligned} \quad (47)$$

Then,

$$v_{xx} = r''(0) \quad \text{and} \quad v_{xx} v_{yy} - v_{xy}^2 = \frac{s(0) r''(0)}{(b - c\gamma)^2} \left((a - b\gamma) h_{xx} + (b - c\gamma) h_{xy} - \frac{h_{xx}^2 s(0)}{r''(0)} \right). \quad (48)$$

Accordingly, since Eq. (44) holds by hypothesis, in order to ensure (46) it is enough to take

$$r''(0) > \frac{h_{xx}^2(\mathbf{0}) s(0)}{[(a - b\gamma) h_{xx} + (b - c\gamma) h_{xy}](\mathbf{0})}. \quad (49)$$

This ends our proof. \square

The next lemma gives a necessary and sufficient condition, in terms of H , for the existence of a function γ that fulfill (44). The proof can be found in the Appendix.

Lemma 14 *There exists a function γ such that (44) holds if and only if*

$$[b h_{xx} + c h_{xy}](\mathbf{0}) \neq 0 \quad \text{or} \quad h_{xx}(\mathbf{0}) > 0. \quad (50)$$

Moreover, in such a case, $\gamma(\mathbf{0})$ can be chosen such that

$$|\gamma(\mathbf{0})| > \left| \left[\frac{a h_{xx} + b h_{xy}}{b h_{xx} + c h_{xy}} \right](\mathbf{0}) \right| \quad \text{and} \quad [\gamma(b h_{xx} + c h_{xy})](\mathbf{0}) < 0 \quad (51)$$

if $[b h_{xx} + c h_{xy}](\mathbf{0}) \neq 0$, and such that

	$h_{xy}(\mathbf{0}) = 0$	$h_{xy}(\mathbf{0}) < 0$	$h_{xy}(\mathbf{0}) > 0$
$b(\mathbf{0}) = 0$	any	$\gamma(\mathbf{0}) > 0$	$\gamma(\mathbf{0}) < 0$
$b(\mathbf{0}) > 0$	$\gamma(\mathbf{0}) < \frac{a(\mathbf{0})}{b(\mathbf{0})}$	$\frac{b(\mathbf{0})}{c(\mathbf{0})} < \gamma(\mathbf{0}) < \frac{a(\mathbf{0})}{b(\mathbf{0})}$	$\gamma(\mathbf{0}) < \min\left(\frac{a(\mathbf{0})}{b(\mathbf{0})}, \frac{b(\mathbf{0})}{c(\mathbf{0})}\right)$
$b(\mathbf{0}) < 0$	$\gamma(\mathbf{0}) > \frac{a(\mathbf{0})}{b(\mathbf{0})}$	$\gamma(\mathbf{0}) > \max\left(\frac{a(\mathbf{0})}{b(\mathbf{0})}, \frac{b(\mathbf{0})}{c(\mathbf{0})}\right)$	$\frac{a(\mathbf{0})}{b(\mathbf{0})} < \gamma(\mathbf{0}) < \frac{b(\mathbf{0})}{c(\mathbf{0})}$

(52)

if $h_{xx}(\mathbf{0}) > 0$. All these conditions are compatible with (43).

Summarizing, if (50) holds, in order to find a solution (δ, γ, v) of (40), (41) and (42), it is enough to take γ satisfying (43) and also (51) or (52), as explained in the last lemma. Thus, we have proved the following.

Theorem 15 *Consider an underactuated system (H, \mathcal{W}) on a 2-manifold Q , with H simple and \mathcal{W} satisfying **W1** and **W2**, and consider a critical point $\alpha_0 = (q_0, 0) \in T^*Q$ of X_H . Fix a local chart adapted to W and centered at q_0 , and consider the corresponding local expression of H given by (17) and (36). If Eq. (50) holds, then (H, \mathcal{W}) is stabilizable at α_0 , i.e. there exists a vector field Y , defined at least around α_0 , such that the closed-loop system defined by $X_H + Y$ is stable at α_0 . Moreover, such a stability can be ensured by the existence of a simple Lyapunov function.*

An analogous result was found first in [14], but in the context of the energy shaping method. The purpose of including this alternative proof here was to introduce the notation, equations and calculations that will be useful later to:

- show a converse⁷ of this result (see Section 3.3),
- and more importantly, to extend the result to asymptotic stabilizability (see Section 3.4).

3.3 A necessary condition for the existence of a simple Lyapunov function

Using the same notation as above, suppose that (H, \mathcal{W}) can be stabilized at α_0 , and that such stabilization is ensured by the existence of a simple Lyapunov function. More precisely, suppose that there exists a vector field Y with image inside \mathcal{W} and a simple Lyapunov function V for $X_H + Y$ and α_0 , defined at least around α_0 . Then, Theorem 7 ensures that Y and V must be given by the simple LCB method. In particular, V must be locally given, in an adapted chart (U, φ) centered at q_0 , by a solution (δ, γ, v) of equations (40), (41) and (42) [and by (18), (36) and (38)]. We want to show from this fact that (50) must be satisfied. To do that, let us consider two cases.

1. [γ satisfies (43)] Define $r(x) := v(x, 0)$, for all x such that $(x, 0) \in U$. Then, as we saw in the previous section, differentiating (41) and evaluating at $\mathbf{0}$ (and using that $\mathbf{0}$ is critical for h), we arrive at Equation (47). Thus, the positivity conditions (46) for v can be studied in terms of (48). From the latter it easily follows that (44) is a necessary condition. But according to Lemma 14, this says that condition (50) must be satisfied.
2. [γ does not satisfy (43)] If $\gamma(\mathbf{0}) = b(\mathbf{0})/c(\mathbf{0})$, then

$$[a - b\gamma](\mathbf{0}) = a(\mathbf{0}) - b(\mathbf{0}) \frac{b(\mathbf{0})}{c(\mathbf{0})} = \frac{\Delta(\mathbf{0})}{c(\mathbf{0})} > 0.$$

On the other hand, if we differentiate (41) and evaluate the result at $\mathbf{0}$, we obtain

$$(a(\mathbf{0}) - b(\mathbf{0})\gamma(\mathbf{0})) v_{xx}(\mathbf{0}) = h_{xx}(\mathbf{0}) \delta(\mathbf{0}).$$

(Recall that $\mathbf{0}$ is critical for v and h). As a consequence, using that $\delta(\mathbf{0}) > 0$ and $v_{xx}(\mathbf{0}) > 0$,

$$h_{xx}(\mathbf{0}) = \frac{[a - b\gamma](\mathbf{0}) v_{xx}(\mathbf{0})}{\delta(\mathbf{0})} > 0.$$

In other words, again, condition (50) must hold.

Combining above discussion with Theorem 15, we have the following characterization.

Theorem 16 *Under the conditions of Theorem 15 (see the first two sentences), (H, \mathcal{W}) is stabilizable at α_0 , and such a stability can be ensured by the existence of a simple Lyapunov function, if and only if (50) holds.*

In Ref. [14], it was shown that the condition⁸ $[b h_{xx} + c h_{xy}](\mathbf{0}) \neq 0$ also implies asymptotic stability (as previously affirmed in [23], without a proof). In any other case, in the same reference, an additional condition is proposed to ensure this kind of stability. We show in the next subsection that no other condition than (50) is needed to this end.

3.4 Asymptotic stabilizability

Fix (H, \mathcal{W}) , q_0 and a local chart (U, φ) as in the previous subsections, and suppose that (50) holds. Following (35), we shall write from now on

$$p_x := p_1 \quad \text{and} \quad p_y := p_2. \tag{53}$$

⁷It can be shown that such a converse is a consequence of combining the necessary condition for shapability given in [14] and the maximality property of the LCB method established in [22] (which implies that shapability is equivalent to the existence of a simple Lyapunov function). Anyway, for the sake of clarity, we prefer to include a direct proof in this paper.

⁸Actually, a weaker condition is considered there (see Theorem III.3).

Let (δ, γ, v) be a solution of (40), (41) and (42), with γ satisfying (43) and with *boundary conditions* given by functions s and r , as described in the proof of Lemma 13. That is to say, δ and v must satisfy

$$\delta(x, 0) = s(x) \quad \text{and} \quad v(x, 0) = r(x), \quad (54)$$

with s and r fulfilling (45) and (49). Let V be given by (18), (36) and (38) (and also (35) and (53)), i.e.

$$V(x, y, p_x, p_y) = \left(p_x^2 \left(\frac{\delta(x, y)}{l(x, y)} - \gamma^2(x, y) \right) + 2\gamma(x, y) p_x p_y + p_y^2 \right) l(x, y) + v(x, y), \quad (55)$$

for some positive function l . Fix μ as in Eq. (16), or equivalently (27). Using (35), (38) and (53), this means that

$$\mu(x, y, p_x, p_y) = \varkappa(\gamma(x, y) p_x + p_y)^2 l^2(x, y).$$

With all these elements, the step 3 of the simple LCB method gives us a control law which, using (23), adopts the form

$$Y(\alpha) = \lambda(\alpha) \left. \frac{\partial}{\partial p_y} \right|_{\alpha}, \quad \forall \alpha \in T^*U, \quad (56)$$

where λ is locally given as [see (15)]

$$\lambda(x, y, p_x, p_y) = -\varkappa(\gamma(x, y) p_x + p_y) l(x, y) - \frac{\{V, H\}(x, y, p_x, p_y)}{(\gamma(x, y) p_x + p_y) l(x, y)}. \quad (57)$$

And according to Theorem 7, the function V given above is a Lyapunov function for $X := X_H|_{T^*U} + Y$ and $\alpha_0 = (q_0, 0)$, with LaSalle submanifold (see Remark 1)

$$\mu^{-1}(0) = \varphi^* (\{(x, y, p_x, p_y) : p_x \gamma(x, y) + p_y = 0\}) \subset T^*U. \quad (58)$$

Below, we are going to show, without any additional assumption (other than (50)), that the boundary conditions s and r [see (54)], and an open subset $T \subset T^*U$ containing α_0 , can be chosen in such a way that the largest X -invariant⁹ submanifold of $\mathcal{S}_0 := \mu^{-1}(0) \cap T$ is the singleton $\{\alpha_0\}$. This would imply, *via* the LaSalle's invariance principle, that α_0 is (locally) asymptotically stable for X (see [24]). The proof will be based on the next two lemmas (the proof of the first one is easy to derive so we omit it for brevity).

Lemma 17 *Given a manifold P , a vector field X on P , a critical point α_0 of X , and a submanifold $\mathcal{S}_0 \subset P$ containing α_0 , let us define*

$$\mathcal{S}_n := \{\alpha \in \mathcal{S}_{n-1} : X(\alpha) \in T\mathcal{S}_{n-1}\}, \quad n \in \mathbb{N}, \quad (59)$$

where we are assuming that each \mathcal{S}_n is a submanifold of \mathcal{S}_{n-1} . Then, the largest X -invariant subset I of \mathcal{S}_0 satisfies

$$\{\alpha_0\} \subset I \subset \bigcap_{n \in \mathbb{N}} \mathcal{S}_n.$$

In particular, if $\mathcal{S}_k = \{\alpha_0\}$ for some $k \in \mathbb{N}$, then $I = \{\alpha_0\}$.

Lemma 18 *There exist boundary conditions s and r , a function γ and an open subset $T \ni \alpha_0$ of T^*U , such that [see (59)]:*

- *the subset \mathcal{S}_1 corresponding to $\mathcal{S}_0 = \mu^{-1}(0) \cap T$ is a submanifold of \mathcal{S}_0 ;*
- *\mathcal{S}_2 is a submanifold of \mathcal{S}_1 ;*
- *$\mathcal{S}_3 = \{\alpha_0\}$.*

⁹Recall that, given a manifold P and a vector field X on P , a subset $S \subset P$ is *X-invariant* if every integral curve of X with initial condition in S is contained in S .

It is enough to take s and r such that [besides (45) and (49)]

$$\frac{s'(0)}{s(0)} \neq - \left[\frac{2(b-c\gamma)}{\Delta} \left(b_x - \gamma c_x - \frac{Bc}{2(b-c\gamma)} \right) \right] (\mathbf{0}), \quad (60)$$

choose $\gamma(\mathbf{0})$ according to (43), (52) and the additional restriction

$$\gamma(\mathbf{0}) \neq \frac{(a(\mathbf{0}), b(\mathbf{0})) \mathbb{M}(\mathbf{0}) \begin{pmatrix} b(\mathbf{0}) \\ c(\mathbf{0}) \end{pmatrix}}{(b(\mathbf{0}), c(\mathbf{0})) \mathbb{M}(\mathbf{0}) \begin{pmatrix} b(\mathbf{0}) \\ c(\mathbf{0}) \end{pmatrix}}, \quad (61)$$

where

$$\mathbb{M} = \begin{bmatrix} r''(0) & \frac{h_{xx}s(0)-(a-b\gamma)r''(0)}{(b-c\gamma)} \\ \frac{h_{xx}s(0)-(a-b\gamma)r''(0)}{(b-c\gamma)} & \frac{h_{xy}s(0)(b-c\gamma)-(a-b\gamma)h_{xx}s(0)+(a-b\gamma)^2r''(0)}{(b-c\gamma)^2} \end{bmatrix}. \quad (62)$$

Proof. According to (58), $\varphi^{*-1}(\mu^{-1}(0))$ can be described as the zero set of the submersion

$$\mathfrak{F}(x, y, p_x, p_y) = \gamma(x, y) p_x + p_y.$$

In the following, we shall omit φ , i.e. we shall identify U and $\varphi(U)$. This means that $q_0 = \mathbf{0}$ and $\alpha_0 = (\mathbf{0}, \mathbf{0})$. We shall proceed in three steps.

1. Let us consider the subset $Z_1 \subset \mu^{-1}(0)$ such that $\mathfrak{F}_*(X)(x, y, p_x, p_y) = 0$, where [see (56)]

$$X = X_H + Y = \frac{\partial H}{\partial p_x} \frac{\partial}{\partial x} + \frac{\partial H}{\partial p_y} \frac{\partial}{\partial y} - \frac{\partial H}{\partial x} \frac{\partial}{\partial p_x} - \left(\frac{\partial H}{\partial y} - \lambda \right) \frac{\partial}{\partial p_y}. \quad (63)$$

That is to say, Z_1 is given by

$$\frac{\partial H}{\partial p_x} \frac{\partial \mathfrak{F}}{\partial x} + \frac{\partial H}{\partial p_y} \frac{\partial \mathfrak{F}}{\partial y} - \frac{\partial H}{\partial x} \frac{\partial \mathfrak{F}}{\partial p_x} - \left(\frac{\partial H}{\partial y} - \lambda \right) \frac{\partial \mathfrak{F}}{\partial p_y} = 0,$$

or equivalently by

$$\frac{\partial H}{\partial p_x} \gamma_x p_x + \frac{\partial H}{\partial p_y} \gamma_y p_x - \frac{\partial H}{\partial x} \gamma - \frac{\partial H}{\partial y} + \lambda = 0, \quad (64)$$

and $p_y = -\gamma(x, y) p_x$. It is easy to see from (17) and (36) that, on $\mu^{-1}(0)$,

$$\frac{\partial H}{\partial x} = \frac{1}{2} B p_x^2 + h_x, \quad \frac{\partial H}{\partial y} = \frac{1}{2} C p_x^2 + h_y, \quad \frac{\partial H}{\partial p_x} = (a - b\gamma) p_x \quad \text{and} \quad \frac{\partial H}{\partial p_y} = (b - c\gamma) p_x, \quad (65)$$

where B is given by (39) and

$$C := a_y - 2\gamma b_y + \gamma^2 c_y.$$

On the other hand, the values of λ on $\mu^{-1}(0)$, according to (57), are

$$\begin{aligned} \lambda(x, y, p_x, -\gamma(x, y) p_x) &= - \lim_{p_y \rightarrow -\gamma p_x} \frac{\{V, H\}(x, y, p_x, p_y)}{(\gamma(x, y) p_x + p_y) l(x, y)} \\ &= - \frac{1}{l(x, y)} \frac{\partial \{V, H\}}{\partial p_y}(x, y, p_x, -\gamma(x, y) p_x). \end{aligned} \quad (66)$$

So, by lengthy, but straightforward calculations, from (55), (65) and (67) we have that

$$\begin{aligned} \lambda(x, y, p_x, -\gamma(x, y) p_x) &= \left[\frac{1}{2} (B\gamma + C) - \gamma_x (a - b\gamma) - \gamma_y (b - c\gamma) + \frac{2(b_x - \gamma c_x) \delta - b \delta_x - c \delta_y}{2l} \right] p_x^2 \\ &\quad + \gamma h_x + h_y - \frac{b v_x + c v_y}{l}. \end{aligned} \quad (67)$$

We are omitting, for simplicity, the evaluation point (x, y) for the functions on the right hand side. Finally, combining (64), (65) and (67), we have at $\mu^{-1}(0)$,

$$\mathfrak{F}_*(X) = \frac{1}{l} \left[(b_x - \gamma c_x) \delta - \frac{b \delta_x + c \delta_y}{2} \right] p_x^2 - \frac{b v_x + c v_y}{l}.$$

Thus, Z_1 is given by the equations

$$\gamma(x, y) p_x + p_y = 0 \quad \text{and} \quad K(x, y) p_x^2 - L(x, y) = 0, \quad (68)$$

where

$$K = (b_x - \gamma c_x) \delta - \frac{b \delta_x + c \delta_y}{2} \quad \text{and} \quad L = b v_x + c v_y. \quad (69)$$

In consequence, Z_1 can be defined by the zero set of the function

$$\mathfrak{G}(x, y, p_x, p_y) = (\gamma(x, y) p_x + p_y, K(x, y) p_x^2 - L(x, y)).$$

We want to see that its related push-forward (omitting the evaluation point of the involved functions)

$$\mathfrak{G}_* = \begin{pmatrix} \gamma_x p_x & \gamma_y p_x & \gamma & 1 \\ K_x p_x^2 - L_x & K_y p_x^2 - L_y & 2 K p_x & 0 \end{pmatrix} \quad (70)$$

has maximal rank around $(\mathbf{0}, \mathbf{0}) = (0, 0, 0, 0)$, i.e. \mathfrak{G} is a submersion when restricted to some open neighborhood of $(\mathbf{0}, \mathbf{0})$. At $(\mathbf{0}, \mathbf{0})$, such a push-forward is given by

$$\mathfrak{G}_{*,(\mathbf{0},\mathbf{0})} = \begin{pmatrix} 0 & 0 & \gamma(\mathbf{0}) & 1 \\ -L_x(\mathbf{0}) & -L_y(\mathbf{0}) & 0 & 0 \end{pmatrix}. \quad (71)$$

Note that the gradient of L can be written

$$\begin{pmatrix} L_x \\ L_y \end{pmatrix} = \begin{pmatrix} v_{xx} & v_{xy} \\ v_{xy} & v_{yy} \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} + \begin{pmatrix} b_x & c_x \\ b_y & c_y \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}. \quad (72)$$

Since $\mathbf{0}$ is critical for v , then, at $\mathbf{0}$,

$$\begin{pmatrix} L_x \\ L_y \end{pmatrix} = \begin{pmatrix} v_{xx} & v_{xy} \\ v_{xy} & v_{yy} \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix}.$$

But we know that the Hessian matrix of v is positive-definite and the function c is always positive. So, the gradient of L can not vanish at $\mathbf{0}$. This implies that (71) has maximal rank at $(\mathbf{0}, \mathbf{0})$. Consequently, there exists an open subset $T_1 \subset T^*U$ containing $(\mathbf{0}, \mathbf{0})$ such that $Z_1 \cap T_1$ is a submanifold of $\mu^{-1}(0) \cap T_1$.

2. Consider now the subset $Z_2 \subset Z_1 \cap T_1$ given by $\mathfrak{G}_*(X)(x, y, p_x, p_y) = 0$. Easy calculations show that Z_2 is given by the points of $Z_1 \cap T_1$ such that

$$\left[(a - b \gamma) (K_x p_x^2 - L_x) + (b - c \gamma) (K_y p_x^2 - L_y) - 2 K \left(\frac{1}{2} B p_x^2 + h_x \right) \right] p_x = 0. \quad (73)$$

We only need to evaluate the second row of \mathfrak{G}_* [see Eq. (70)] to the components of X [see (63), (65) and (67)]. In the following, we assume that $K(\mathbf{0}) \neq 0$. Observe that, since $\delta(x, 0) = s(x)$, we have $\delta_x(\mathbf{0}) = s'(0)$, and using the kinetic equation (41) at the origin

$$\delta_y(\mathbf{0}) = \left[\frac{B s - (a - b \gamma) s'(0)}{b - c \gamma} \right] (\mathbf{0}).$$

So [see (69)]

$$K(\mathbf{0}) = \left[\left(b_x - \gamma c_x - \frac{B c}{2(b - c \gamma)} \right) s(0) + \frac{1}{2} \left(\frac{\Delta}{b - c \gamma} \right) s'(0) \right] (\mathbf{0}),$$

and consequently, the condition $K(\mathbf{0}) \neq 0$ gives precisely Eq. (60). Under such an assumption, we can replace p_x^2 by $\frac{L}{K}$ in (73) [see (68)], and we get

$$\left[(a - b\gamma) \left(K_x \frac{L}{K} - L_x \right) + (b - c\gamma) \left(K_y \frac{L}{K} - L_y \right) - 2K \left(\frac{1}{2} B \frac{L}{K} + h_x \right) \right] p_x = 0, \quad (74)$$

on some open subset $U' \subset U$ containing $\mathbf{0}$ (where K is non vanishing). Moreover, since $L(\mathbf{0}) = 0$ [see (69)] and $h_x(\mathbf{0}) = 0$, we have at $\mathbf{0}$ that

$$\frac{L_x K - L K_x}{K^2} = \frac{L_x}{K}, \quad \frac{L_y K - L K_y}{K^2} = \frac{L_y}{K},$$

and then, the bracketed expression in (74) takes the following form at $\mathbf{0}$

$$(a - b\gamma) L_x + (b - c\gamma) L_y.$$

Using (72), this in turn may be written as

$$(a, b) \begin{bmatrix} v_{xx} & v_{xy} \\ v_{xy} & v_{yy} \end{bmatrix} \begin{pmatrix} b \\ c \end{pmatrix} - \gamma (b, c) \begin{bmatrix} v_{xx} & v_{xy} \\ v_{xy} & v_{yy} \end{bmatrix} \begin{pmatrix} b \\ c \end{pmatrix}.$$

Then, if we assume that, at $\mathbf{0}$,

$$\gamma \neq \frac{(a, b) \begin{bmatrix} v_{xx} & v_{xy} \\ v_{xy} & v_{yy} \end{bmatrix} \begin{pmatrix} b \\ c \end{pmatrix}}{(b, c) \begin{bmatrix} v_{xx} & v_{xy} \\ v_{xy} & v_{yy} \end{bmatrix} \begin{pmatrix} b \\ c \end{pmatrix}},$$

it follows that (74) will hold only if $p_x = 0$ around $(x, y) = \mathbf{0}$. It is worth mentioning that this condition is compatible with (43) and (52). Note that, using Eq. (47), the condition above is given precisely by (61) and (62). In conclusion, there exists an open neighborhood $T'_2 := \pi^{-1}(U') \subset T^*U$ [which contains the point $(\mathbf{0}, \mathbf{0})$] such that the subset $Z_2 \cap T'_2$ is given by

$$\gamma p_x + p_y = 0, \quad K p_x^2 - L = 0, \quad p_x = 0,$$

or equivalently

$$p_x = p_y = L = 0. \quad (75)$$

This means that $Z_2 \cap T'_2$ can be described by as the zero set of the function

$$\mathfrak{H}(x, y, p_x, p_y) = (p_y, p_x, L(x, y)).$$

The push-forward of \mathfrak{H} at $(\mathbf{0}, \mathbf{0})$ is given by

$$\mathfrak{H}_{*,(\mathbf{0},\mathbf{0})} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ L_x(\mathbf{0}) & L_y(\mathbf{0}) & 0 & 0 \end{pmatrix}.$$

Again, since $L_x(\mathbf{0})$ and $L_y(\mathbf{0})$ cannot be both zero, we conclude that $\mathfrak{H}_{*,(\mathbf{0},\mathbf{0})}$ has maximum rank. Thus, there exists inside T'_2 an open neighborhood T_2 of $(\mathbf{0}, \mathbf{0})$ such that $Z_2 \cap T_2$ is a submanifold of $Z_1 \cap T_1 \cap T_2$.

- Now, consider the subset $Z_3 \subset Z_2 \cap T_2$ defined by $\mathfrak{H}_*(X)(x, y, p_x, p_y) = 0$. Using (63), (65) and (67), it follows that, along $Z_2 \cap T_2$ [see (75)]

$$X = -h_x \frac{\partial}{\partial p_x} + \gamma h_x \frac{\partial}{\partial p_y},$$

so, in order for $\mathfrak{H}_*(X)$ to vanish, it is necessary that $h_x = 0$. But, if this is the case, using the potential equations

$$(a - b\gamma)v_x + (b - c\gamma)v_y = \delta h_x,$$

or equivalently, $a v_x + b v_y - \gamma L = \delta h_x$, we have on Z_3 that

$$L = b v_x + c v_y = 0 \quad \text{and} \quad a v_x + b v_y = 0,$$

i.e.

$$\mathbb{H} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = 0.$$

This is possible if and only if all the points of Z_3 are critical for v . By the *Morse Lemma*, since $\mathbf{0}$ is a non-degenerate critical point of v , there exists a neighborhood $U'' \subset U$ of $\mathbf{0}$ such that $\pi(Z_3) \cap U'' = \{\mathbf{0}\}$.

But $p_x = p_y = 0$ on Z_3 , which implies that $Z_3 \cap T_3 = \{(\mathbf{0}, \mathbf{0})\} = \{\alpha_0\}$ for $T_3 := \pi^{-1}(U'')$.

Summing up, if we define $T := T_1 \cap T_2 \cap T_3$ and $\mathcal{S}_0 := \mu^{-1}(0) \cap T$, from (59) we obtain $\mathcal{S}_1 = Z_1 \cap T$, which is a submanifold of \mathcal{S}_0 , $\mathcal{S}_2 = Z_2 \cap T$, which is a submanifold of \mathcal{S}_1 , and $\mathcal{S}_3 = \{\alpha_0\}$. Hence, the three points of the lemma follows. \square

Concluding, if Eq. (50) holds, asymptotic stability is ensured. Reciprocally, if we can ensure asymptotic stabilizability by the existence of a simple Lyapunov function, then we can also ensure stabilizability, and Theorem 16 implies that (50) holds. In other terms,

Theorem 19 *Under the conditions of Theorem 15 (see the first two sentences), (H, \mathcal{W}) is asymptotically stabilizable at α_0 , and such a stability can be ensured by the existence of a simple Lyapunov function, if and only if H satisfies Eq. (50).*

3.5 Example: the inertia wheel pendulum

Now, we illustrate our results with a concrete underactuated system (H, \mathcal{W}) with two degrees of freedom, the *inertia wheel pendulum*:

- the configuration is $Q = S^1 \times S^1$, whose natural almost-global coordinates will be denoted (θ, ψ) ;
- the Hamiltonian is

$$H(\theta, \psi, p_\theta, p_\psi) = \frac{1}{2} (p_\theta, p_\psi) \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{pmatrix} p_\theta \\ p_\psi \end{pmatrix} + M(1 + \cos \theta),$$

where a, b, c, M are constants and $a, b, M, ac - b^2$ are strictly greater than zero;

- and the space of actuators is given by the subbundle $W = \langle d\psi \rangle$.

We shall find, by using the simple LCB method, a control law Y for this system and a related simple Lyapunov function V which make the closed-loop system $X_H + Y$ asymptotically stable at $(\theta, \psi, p_\theta, p_\psi) = (0, 0, 0, 0) = (\mathbf{0}, \mathbf{0})$.

Let us go back to Section 3.1, and replace x by θ and y by ψ . The Eq. (50) in this case says that (because $h_{\theta\psi}(\mathbf{0}) = 0$)

$$b h_{\theta\theta}(\mathbf{0}) \neq 0 \quad \text{or} \quad h_{\theta\theta}(\mathbf{0}) > 0,$$

which is equivalent to $h_{\theta\theta}(\mathbf{0}) \neq 0$, since $b \neq 0$. And it does hold, because $h_{\theta\theta}(\mathbf{0}) = -M \neq 0$. Then, the inertia wheel pendulum can be asymptotically stabilized around $(\mathbf{0}, \mathbf{0})$, as it is well-known. On the other hand, according to (39), we have that $B = 0$. So, the kinetic and potential equations read [see (40) and (41)]

$$(a - b\gamma)\delta_\theta + (b - c\gamma)\delta_\psi = 0, \tag{76}$$

and

$$(a - b\gamma) v_\theta + (b - c\gamma) v_\psi = -M \delta \sin \theta, \quad (77)$$

respectively. Let us construct a solution (δ, γ, v) of above equations, with $\delta > 0$ and v positive-definite w.r.t. $\mathbf{0}$. We shall take γ constant. Following the steps of Section 3.2, it is enough to take γ such that $\gamma \neq b/c$ [see (43)] and, using Eq. (51) of Lemma 14 (since $h_{\theta\theta}(\mathbf{0}) = -M < 0$), also ask that

$$|\gamma| > \frac{a}{b} \quad \text{and} \quad -\gamma b M < 0.$$

The second inequality says that γ is positive, so, above equations only impose the condition $\gamma > a/b$. Note also that, since $a, b, c, ac - b^2 > 0$, we have that $a/b > b/c$. Hence, all the conditions on γ reduce to

$$\gamma > \frac{a}{b}.$$

Remark 20 *Note that, for this system, we can not take $\gamma = b/c$. In fact, in such a case, according to the calculations we made in Section 3.3, the positivity of δ and v would impose that $h_{\theta\theta}(\mathbf{0}) > 0$, which is not true.*

Regarding the boundary conditions defining δ and v , i.e. the functions s and r , respectively, we must ask [see (45) and (49)]

$$s(0) > 0, \quad r(0) = r'(0) = 0 \quad \text{and} \quad r''(0) > \frac{h_{\theta\theta}^2(\mathbf{0}) s(0)}{(a - b\gamma) h_{\theta\theta}(\mathbf{0})} = -\frac{M s(0)}{a - b\gamma}. \quad (78)$$

And to ensure asymptotic stabilizability, according to (60), (61) and (62) of Lemma 18,

$$\frac{s'(0)}{s(0)} \neq 0, \quad \text{i.e.} \quad s'(0) \neq 0,$$

and

$$\gamma \neq \frac{\eta^2 ab r''(0) - \eta [M s(0) + \zeta r''(0)] (ac + b^2) + \zeta M s(0) + \zeta^2 r''(0) bc}{\eta^2 b^2 r''(0) - 2\eta [M s(0) + \zeta r''(0)] bc + \zeta M s(0) + \zeta^2 r''(0) c^2}, \quad (79)$$

where $\zeta := a - b\gamma$ and $\eta := b - c\gamma$. Thus, take any number $\gamma > a/b$, any function s such that $s(0) > 0$ and $s'(0) \neq 0$, and any function¹⁰ r such that $r(0) = r'(0) = 0$ and $r''(0)$ satisfying (78) and (79), and let us apply the Method of Characteristics to Eqs. (76) and (77), with boundary conditions on $\psi = 0$ given by s and r . The characteristic equations for (76) are

$$\begin{aligned} \dot{\theta} &= a - b\gamma, & \theta(0) &= \theta_0, \\ \dot{\psi} &= b - c\gamma, & \psi(0) &= 0, \\ \dot{\delta} &= 0, & \delta(0) &= s(\theta_0). \end{aligned}$$

Then

$$\theta(t) = (a - b\gamma)t + \theta_0, \quad \psi(t) = (b - c\gamma)t,$$

and defining $\Upsilon := (a - b\gamma) / (b - c\gamma)$ we find

$$\delta(\theta, \psi) = s(\theta - \Upsilon \psi). \quad (80)$$

The characteristic equation for (77) (and for δ given above) is $\dot{v} = -M s(\theta_0) \sin((a - b\gamma)t + \theta_0)$, and integrating we obtain

$$v(\theta, \psi) = \frac{M s(\theta - \Upsilon \psi)}{a - b\gamma} (\cos \theta - \cos(\theta - \Upsilon \psi)) + r(\theta - \Upsilon \psi). \quad (81)$$

Finally, with δ and v given by (80) and (81), and considering any positive function l , we have from Eqs. (55), (56) and (57) the control law Y and the Lyapunov function V we are looking for.

¹⁰Additionally, the functions s and r may be taken with period 2π in order to look for a quasi-global solution.

4 Appendix

Let us prove Lemma 14. Omitting the evaluation point $\mathbf{0}$, suppose first that $h_{xx} \leq 0$ and $b h_{xx} + c h_{xy} = 0$. Then,

$$(a - b\gamma) h_{xx} + (b - c\gamma) h_{xy} = a h_{xx} + b h_{xy} - \gamma (b h_{xx} + c h_{xy}) = \left(a - \frac{b^2}{c}\right) h_{xx} = \frac{\Delta}{c} h_{xx} \leq 0$$

for all γ . This proves the first implication of the lemma. For the converse, suppose first that $b h_{xx} + c h_{xy} \neq 0$. Since

$$(a - b\gamma) h_{xx} + (b - c\gamma) h_{xy} = a h_{xx} + b h_{xy} - \gamma (b h_{xx} + c h_{xy}),$$

for any γ with sign opposite to $b h_{xx} + c h_{xy}$ and such that

$$|\gamma| > \left| \frac{a h_{xx} + b h_{xy}}{b h_{xx} + c h_{xy}} \right|,$$

equation (44) holds. This implies Eq. (51). Now, suppose that $h_{xx} > 0$. If $h_{xy} = 0$, since $a > 0$, it is clear that it is enough to choose γ such that $a > b\gamma$. If instead $h_{xy} \neq 0$, we distinguish three cases: $b = 0$, $b > 0$ and $b < 0$.

- If $b = 0$, then $a - b\gamma = a > 0$ and $b - c\gamma = -c\gamma$, and consequently it is sufficient to choose γ with opposite sign to h_{xy} .
- If $b > 0$, we show that it is possible to take γ so as to fulfill one of the following expressions

$$a - b\gamma > 0 \quad \text{and} \quad b - c\gamma > 0,$$

or

$$a - b\gamma > 0 \quad \text{and} \quad b - c\gamma < 0.$$

In order to make $a - b\gamma > 0$, we need $\gamma < \frac{a}{b}$. If in addition $b - c\gamma > 0$, then $\gamma < \frac{b}{c}$. Hence, it suffices to take $\gamma < \min\left(\frac{a}{b}, \frac{b}{c}\right)$. On the contrary, if $b - c\gamma < 0$, then we can take $\frac{b}{c} < \gamma < \frac{a}{b}$, which is always possible because

$$ac - b^2 > 0 \quad \text{and} \quad b > 0 \quad \Rightarrow \quad \frac{b}{c} < \frac{a}{b}.$$

Thus, if $h_{xy} > 0$ we choose $\gamma < \min\left(\frac{a}{b}, \frac{b}{c}\right)$ and if $h_{xy} < 0$ we take $\frac{b}{c} < \gamma < \frac{a}{b}$. In both cases we get the desired result.

- If $b < 0$, then $a - b\gamma > 0$ implies $\gamma > \frac{a}{b}$. If in addition $b - c\gamma > 0$, then $\gamma < \frac{b}{c}$ and so we can take γ such that $\frac{a}{b} < \gamma < \frac{b}{c}$, which is always possible because

$$ac - b^2 > 0 \quad \text{and} \quad b < 0 \quad \Rightarrow \quad \frac{a}{b} < \gamma < \frac{b}{c}.$$

On the contrary, if $b - c\gamma < 0$, then $\gamma > \frac{b}{c}$ and it is sufficient to choose $\gamma > \max\left(\frac{a}{b}, \frac{b}{c}\right)$. Again, both cases lead to the desired result. All these alternatives give the table (52).

The last assertion of the lemma is immediate, because all conditions on γ are inequalities.

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